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Roll No. | | | | | | | | | | | |

2019

S. No. of Question Paper : **2249**

Unique Paper Code : **32351202** **IC**

Name of the Paper : **Differential Equations**

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : **II**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Use of non-programmable scientific calculators is allowed.

Section–I

1. Attempt any *three* parts. Each part is of **5** marks.

(*a*) Solve the initial value problem :

$$(x^2+1)\frac{dy}{dx}+4xy=x,\ y(2)=1.$$

(*b*) Determine the constant A in the following equation such that the equation is exact, and solve the resulting exact equation :

$$(x^2+3xy)dx+(Ax^2+4y)dy=0.$$

P.T.O.

(*c*) Solve the differential equation :

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=4x.$$

(*d*) Solve the differential equation :

$$2xy\frac{dy}{dx}=x^2+2y^2.$$

2. Attempt any *two* parts. Each part is of **5** marks.

(*a*) A cylindrical tank with length 5 ft and radius 3 ft is situated with its axis horizontal. If a circular bottom hole with a radius of 1 inch is opened and the tank is initially half full of xylene, how long will it take for the liquid to drain completely ?

(*b*) Suppose that sodium pentobarbital is used to anesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 milligrams (mg) of sodium pentobarbital per kilogram of the dog's body weight. Suppose also that sodium pentobarbital is eliminated exponentially from the dog's bloodstream, with a half life of 5 hours. What single dose should be administered in order to anesthetize a 50 kilogram dog for 1 hour ?

(c) Suppose that a motorboat is moving at 40 ft/sec when its motor suddenly quits, and that 10 seconds later the boat has slowed to 20 ft/sec. Assume that the resistance it encounters is proportional to its velocity. How far will the boat cast in all ?

Section–II

3. Attempt any *two* parts. Each part is of **7.5** marks.

(a) Consider the American system of two lakes : Lake Erie feeding into Lake Ontario. Assuming that volume in each lake to remain constant and that Lake Erie is the only source of pollution for Lake Ontario.

(i) Write down a differential equation describing the concentration of pollution in each of two lakes, using the variables V for volume, F for flow, $c(t)$ for concentration at time t and subscripts 1 for Lake Erie and 2 for Lake Ontario.

(ii) Suppose that only unpolluted water flows into Lake Erie. How does this change the model proposed ?

(iii) Solve the system of equations to get expression for the pollution concentration $c_1(t)$ and $c_2(t)$.

(*b*) In view of the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h_0 X.$$

(*i*) Find the non-zero equilibrium population.

(*ii*) At what critical harvesting rate can extinction occur ?

(*c*) Consider the population of the country. Assume constant per capita birth and death rates and that the population follows an exponential growth (or decay) process. Assume there to be significant immigration and emigration of people into and out of the country.

(*i*) Assuming the overall immigration and emigration rates are constant, formulate a single differential equation to describe the population size over time.

(*ii*) Suppose instead that all immigration and emigration occurs with a neighbouring country, such that the net movement from one country to the another is proportional to the population difference between the two countries and such that people move to the country with the larger population. Formulate a coupled system of equations as a model for this situation.

Section–III

4. Attempt any *four* parts. Each part is of 5 marks.

(*a*) Find general solutions (for $x > 0$) of the Euler's equation :

$$x^2 y'' + 7xy' + 25y = 0.$$

(*b*) Solve the initial value problem by using the method of undetermined coefficients :

$$y'' + y = \sin x ; \quad y(0) = 0, y'(0) = -1.$$

(*c*) Use the method of variation of parameters to find the solution of the differential equation :

$$y'' + 3y' + 2y = 4e^x.$$

(d) A mass of 3 kg is attached to the end of a spring that is stretched 20 cm by a force of 15 N. It is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -10$ m/s. Find the amplitude, period, and frequency of the resulting motion.

(e) A body of mass $m = 2$ kg is attached to both a spring with a spring constant $k = 4$ and a dashpot with a damping constant $c = 3$. The mass is set in motion with initial position $x_0 = 2$ and initial velocity $v_0 = 0$. Find the position function $x(t)$ and determine whether the motion is overdamped, critically damped or underdamped. If it is underdamped, find its pseudofrequency, pseudoperiod of oscillation and its time varying amplitude.

Section–IV

5. Attempt any *two* parts. Each part is of **7.5** marks.

(a) Consider a simple model for a battle between two armies. Assumed that the probability of a single bullet hitting its target is constant. Suppose that the soldiers from the red army are visible to the blue army. But the soldiers from the blue army are hidden.

(*i*) Develop the model for describing the rate of change of number of soldiers in each of the armies.

(*ii*) By making appropriate assumptions, extend the model to include the reinforcements if both of the armies receive reinforcements at constant rates.

(*b*) Consider a disease where all those who are infected remain contagious for life. Assume that there are no births and deaths.

(*i*) Write down suitable word equations for the rate of change of number Susceptible and Infective and hence develop a pair of differential equations.

(*ii*) Use the chain rule to find a relationship between the number of susceptibles and the number of infectives.

(*iii*) Draw a sketch of typical phase-plane trajectories. Deduce the direction of travel along the trajectories providing reasons.

(*c*) A model of a three species interaction is :

$$\frac{dX}{dt} = a_1X - b_1XY - c_1XZ,$$

$$\frac{dY}{dt} = a_2XY - b_2Y,$$

$$\frac{dZ}{dt} = a_3XZ - b_3Z.$$

Where a_i b_i c_i for i = 1, 2, 3 are all positive constants. Here X(t) is the prey density and Y(t) and Z(t) are the two predator species densities.

(*i*) Find all possible equilibrium populations. Is it possible for all three populations to coexist in equilibrium ?

(*ii*) What does this suggest about introducing an additional predator into an ecosystem ?